# Getting Ready to Teach Unit 3

## Learning Path in the Common Core Standards

In Grade 4, students connected fractions with addition and multiplication: $\frac{3}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3 \times \frac{1}{2}$. In this unit, students connect fractions with division: $5 \div 3 = \frac{5}{3}$.

Students use their understanding of multiplication by a fraction to generalize a formula for the product of two fractions. They connect their understanding of division as equal sharing to fractions.

Visual models and real world situations are used throughout the unit to illustrate important fraction concepts.

*Math Expressions* **VOCABULARY**

As you teach this unit, emphasize understanding of these terms.

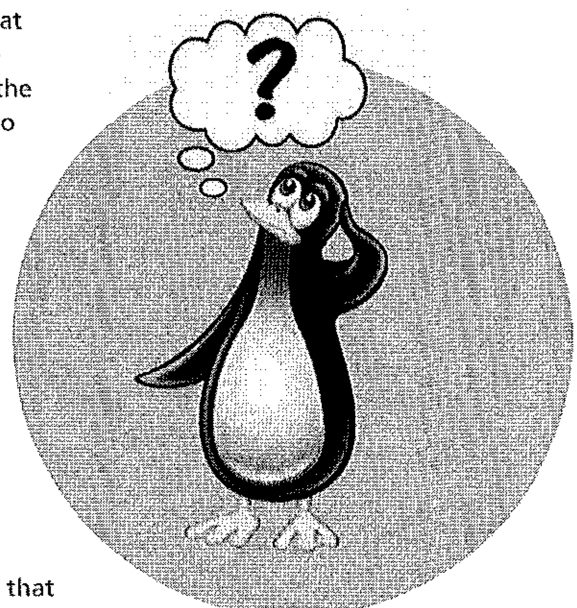
- unsimplify
- *n*-split

*See the Teacher Glossary*

## Help Students Avoid Common Errors

*Math Expressions* gives students opportunities to analyze and correct errors, explaining why the reasoning was flawed.

In this unit, we use Puzzled Penguin to show typical errors that students make. Students enjoy teaching Puzzled Penguin the correct way, and explaining why this way is correct and why the error is wrong. The following common errors are presented to students as letters from Puzzled Penguin and as problems in the Teacher Edition that were solved incorrectly by Puzzled Penguin:




- **Lesson 5:** Simplifying a product by dividing the denominators of both factors by the same number
- **Lesson 6:** Multiplying mixed numbers by multiplying whole number parts and fraction parts separately
- **Lesson 8:** Adding numerators and adding denominators to find a fraction sum
- **Lesson 12:** Finding $\frac{1}{2}$ of a quantity by dividing it by $\frac{1}{2}$

In addition to Puzzled Penguin, there are other suggestions listed in the Teacher Edition to help you watch for situations that may lead to common errors. As a part of the Unit Test Teacher Edition pages, you will find a common error and prescription listed for each test item.

## Multiply a Whole Number by a Fraction

Lessons 1 2 3

**Meaning of Multiplication** Students begin by multiplying a whole number by a fraction. In Grade 4, students interpreted multiplication of a fraction by a whole number as repeated addition.

$$3 \cdot \frac{2}{5} = \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{6}{5} = 1\frac{1}{5}$$

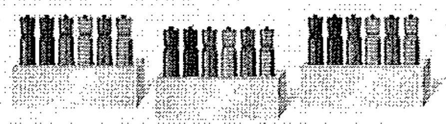
In Grade 5, they interpret multiplication of a whole number by a fraction as finding that fraction of the whole number.

**Multiplying by a Unit Fraction** In Lesson 1, students explore cases in which the product of a unit fraction and a whole number is a whole number. They relate multiplication by a fraction to the equal groups meaning of multiplication.

3. Markers come in boxes of 6. Alta has 3 boxes.

6 taken 3 times = 18 markers

$3 \cdot 6 =$ 18 markers

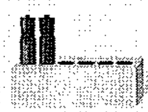


3 groups of 6

4. Isabel has $\frac{1}{3}$ of a box of 6 markers.

6 taken $\frac{1}{3}$ time = 2 markers

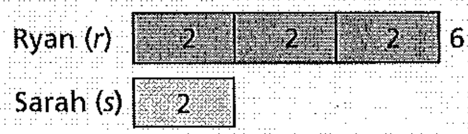
$\frac{1}{3} \cdot 6 =$ 2 markers



$\frac{1}{3}$ group of 6

Because $\frac{1}{3}$ is 1 of 3 equal parts, finding $\frac{1}{3} \cdot 6$, or $\frac{1}{3}$ *of* 6, requires dividing 6 into 3 equal parts.

**Multiplicative Comparisons** In Lesson 1, students also solve multiplicative comparison problems, representing a comparison with comparison bars and with multiplication and division equations. They see that if a quantity $b$ is $n$ times a quantity $a$, then $a$ is $\frac{1}{n}$ times $b$. For example, 6 is 3 times 2, so 2 is $\frac{1}{3}$ times 6.

> To prepare for a family gathering, Sara and Ryan made soup. Sara made 2 quarts. Ryan made 6 quarts.
>
> You can compare amounts using multiplication and division.
>
> Let $r$ equal the number of quarts Ryan made.
> Let $s$ equal the number of quarts Sara made.
>
> Ryan made 3 times as many quarts as Sara.
>
> $r = 3 \cdot s$
>
> 
> 
>
> Sara made one third as many quarts as Ryan.
>
> $s = \frac{1}{3} \cdot r$ or $s = r \div 3$

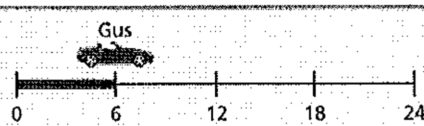
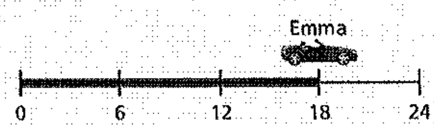
### *from* THE PROGRESSIONS FOR THE COMMON CORE STATE STANDARDS ON NUMBER AND OPERATIONS—FRACTIONS

**Multiplication by a Unit Fraction** With their new understanding of the connection between fractions and division, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}$$

This in turn extends to multiplication of any quantity by a fraction. Just as $\frac{1}{3} \times 5$ is one part when 5 is partitioned into 3 parts, so $\frac{4}{3} \times 5$ is 4 parts when 5 is partitioned into 3 parts.

**Multiplying by a Non-Unit Fraction** In Lesson 2, students explore multiplication of a whole number by a non-unit fraction in cases where the product is a whole number. They see that $\frac{a}{b} \cdot n$ is the same as $a \cdot (\frac{1}{b} \cdot n)$. For example, $\frac{3}{4} \cdot 24 = 3 \cdot (\frac{1}{4} \cdot 24)$. In other words, to find $\frac{3}{4}$ of 24, calculate $\frac{1}{4}$ of 24 and multiply the result by 3.

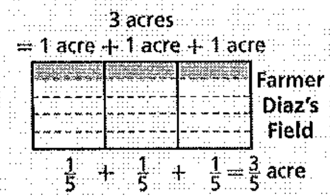
> Silver City is 24 miles away. Gus has driven $\frac{1}{4}$ of the distance. Emma has driven $\frac{3}{4}$ of the distance.
>
> 
> 
>
> 1. How many miles has Gus driven? 6 mi
> 2. How many miles has Emma driven? 18 mi
>
> 
> 
>
> 3. How many times as far as Gus has Emma driven? 3 times as far

**Fractional Products** Lesson 3 extends students' understanding of multiplication of a whole number by a fraction to cases where the product is a fraction. As shown below, a real world situation is used to demonstrate that we can find a unit fraction of a whole number by finding that fraction of each 1 whole and then adding the results.

Farmer Diaz has 3 acres of land. He plows $\frac{1}{5}$ of this land. The number of acres he plows is

$\frac{1}{5}$ of 3 or $\frac{1}{5} \cdot 3$

The diagram at the right shows Farmer Diaz's land divided vertically into 3 acres. The dashed horizontal segments divide the land into five parts. The shaded strip is the $\frac{1}{5}$ of the land Farmer Diaz plowed.



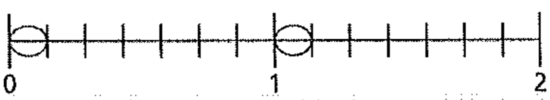

The drawing shows that taking $\frac{1}{5}$ of the 3 acres is the same as taking $\frac{1}{5}$ of each acre and combining them. We can show this mathematically.

$$\begin{aligned}\frac{1}{5} \cdot 3 &= \frac{1}{5}(1 + 1 + 1)\\ &= \frac{1}{5} \cdot 1 + \frac{1}{5} \cdot 1 + \frac{1}{5} \cdot 1\\ &= \frac{1}{5} + \frac{1}{5} + \frac{1}{5}\\ &= \frac{3}{5}\end{aligned}$$

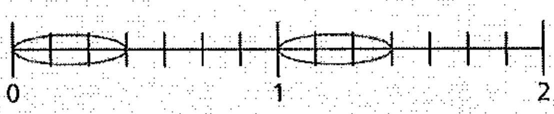
So, $\frac{1}{5}$ of the 3 acres is $\frac{3}{5}$ acre.

Number lines are used to extend this idea to non-unit fractions: To find a fraction of a whole number, find that fraction of each 1 whole and add.

$\frac{1}{7} \cdot 2 = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$




$\frac{3}{7} \cdot 2 = \frac{3}{7} + \frac{3}{7} = \frac{6}{7}$

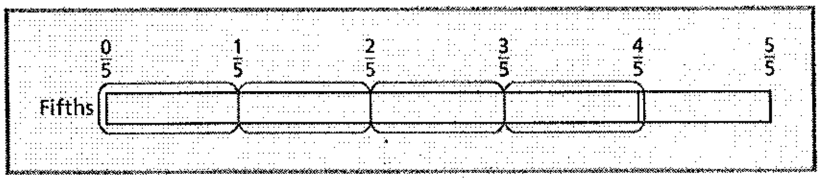


## Multiply a Fraction by a Fraction

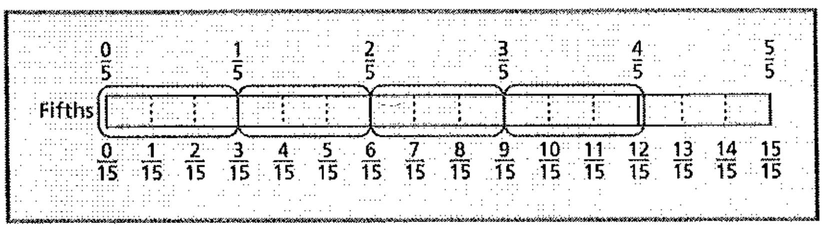
Lessons 4 5 6

**Fraction-Bar Model** Lesson 4 uses a fraction-bar model and an area model to illustrate multiplying two fractions. Both models help students understand why the product of two fractions is the product of the numerators over the product of the denominators.

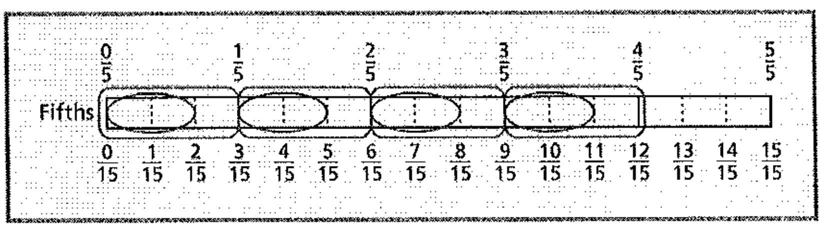
To model $\frac{2}{3} \cdot \frac{4}{5}$, or $\frac{2}{3}$ *of* $\frac{4}{5}$, first circle four of the fifths on the fifths bar. To find the product, we must find $\frac{2}{3}$ of each of the circled fifths.




Divide each fifth into thirds, which divides the whole bar into fifteenths.




Circle $\frac{2}{3}$ of each of the circled fifths. Each of these circled groups is 2 fifteenths of the whole bar.




We have circled 4 groups of 2 fifteenths or $\frac{8}{15}$. So $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$. The product is the product of the numerators over the product of the denominators.

**Relating the Model to the Formula** The pattern shown in the bar model for $\frac{2}{3} \cdot \frac{4}{5}$ can be generalized to any product $\frac{a}{b} \cdot \frac{c}{d}$.

- To find the denominator: $b$-split each $\frac{1}{d}$.

  $\frac{a}{b} \cdot \frac{c}{d} = \frac{}{b \cdot d}$ $b \cdot d$ is the new denominator.
- To find the numerator: take $c$ groups of $a$ of the new unit fractions.

  $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ $a \cdot c$ is the new numerator.

*from* THE PROGRESSIONS FOR THE COMMON CORE STATE STANDARDS ON NUMBER AND OPERATIONS—FRACTIONS

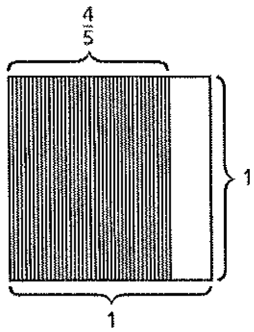
**Multiplying Two Fractions** Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

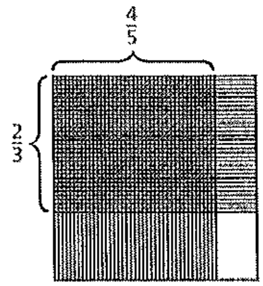
for whole numbers $a$, $b$, $c$, $d$, with $b$, $d$ not zero. Grade 5 students need not express the formula in this general algebraic form, but rather reason out many examples using fraction strips and number line diagrams.

**Area Model** The product $\frac{2}{3} \cdot \frac{4}{5}$ can also be modeled as the area of a rectangle with side lengths $\frac{2}{3}$ and $\frac{4}{5}$. To draw such a rectangle, start with a unit square.

Divide the square vertically into fifths. Shade $\frac{4}{5}$.




Divide the square horizontally into thirds, and shade $\frac{2}{3}$.




$\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$

The area where the shading overlaps is a $\frac{2}{3}$-unit by $\frac{4}{5}$-unit rectangle. It is made up of 8 small rectangles, each of which represents $\frac{1}{15}$ square unit, so $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$.

**Relating the Model to the Formula** The pattern shown in the area model for $\frac{2}{3} \cdot \frac{4}{5}$ can be generalized to any product $\frac{a}{b} \cdot \frac{c}{d}$.

- To find the denominator: The unit square is divided into $b$ parts one way and $d$ parts the other way. There are $b \cdot d$ equal parts in the whole unit square.

  $\frac{a}{b} \cdot \frac{c}{d} = \frac{\quad}{b \cdot d}$ $\quad$ $b \cdot d$ is the new denominator.

- To find the numerator: $a$ parts are shaded one way and $c$ parts the other way, so $a \cdot c$ parts are in the double-shaded $\frac{a}{b}$ by $\frac{c}{d}$ rectangle.

  $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ $\quad$ $a \cdot c$ is the new numerator.

**Simplifying before Multiplying** Lesson 5 introduces the idea of simplifying—that is, dividing numerators and denominators by common factors—before multiplying. For example, students can find $\frac{9}{10} \cdot \frac{5}{12}$ as shown:

$$\frac{{}^{3}\cancel{9}}{{}_{2}\cancel{10}} \cdot \frac{\cancel{5}^{1}}{\cancel{12}_{4}} = \frac{3}{8}$$

**Multiplying with Mixed Numbers** Lesson 6 extends what students have learned about multiplying with fractions to multiplying with mixed numbers. After exploring area models, students generalize a method for multiplying with mixed numbers: write both factors as fractions and then multiply the numerators and multiply the denominators.

*from* **THE PROGRESSIONS FOR THE COMMON CORE STATE STANDARDS ON NUMBER AND OPERATIONS—FRACTIONS**

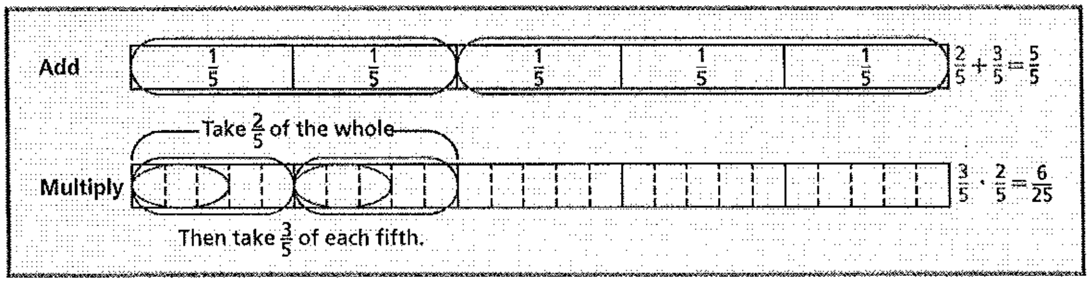
**Using an Area Model** For more complicated examples, an area model is useful, in which students work with a rectangle that has fractional side lengths, dividing it up into rectangles whose sides are the corresponding unit fractions.

## Operations with Fractions and Whole Numbers

Lessons **7** **8**

**Relating Operations** Students relate multiplication with fractions to other fraction operations, and they relate multiplication with fractions to multiplication with whole numbers.

- If we start with a fraction and add another fraction, the result is always greater than the original fraction. If we start with a fraction and multiply by a fraction, the result is always less than the original fraction.




- When we multiply two whole numbers greater than 1, the result is always greater than either number. When we multiply two fractions less than 1, the result is always less than either fraction.

  $3 \cdot 4 = 12$ $\quad$ $12 > 3$ and $12 > 4$

  $\frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}$ $\quad$ $\frac{4}{15} < \frac{1}{3}$ and $\frac{4}{15} < \frac{4}{5}$

- To add and subtract fractions, we need to find a common denominator. To multiply fractions, a common denominator is not needed.

In Lessons 7 and 8, students solve real world problems like the following, in which they need to determine which operation is required.

**Write an equation. Then solve.**

Equations may vary. Possible equations are given.

30. Daniel puts some wheat flour into an empty bowl. Then he adds $\frac{2}{3}$ cup of rye flour to make a total of $2\frac{5}{12}$ cups of flour. How much wheat flour is in the bowl?

    $x + \frac{2}{3} = 2\frac{5}{12}$; $1\frac{3}{4}$ cups

31. Mañuela has a bag containing $5\frac{1}{3}$ cups of sugar. She uses $\frac{1}{8}$ of the sugar in a recipe. How much sugar does she use?

    $x = \frac{1}{8} \cdot 5\frac{1}{3}$; $\frac{2}{3}$ cup

**Properties of Multiplication** Students learn that the Commutative and Associative Properties of Multiplication and the Distributive Property of Multiplication over Addition are true for fractions. For the Commutative and Associative Properties, they explain the steps in algebraic proofs of the properties.

**Commutative Property of Multiplication**

$\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$

Look at the proof of the Commutative Property below.

$$\frac{a}{b} \cdot \frac{c}{d} \underset{\text{Step 1}}{=} \frac{a \cdot c}{b \cdot d} \underset{\text{Step 2}}{=} \frac{c \cdot a}{d \cdot b} \underset{\text{Step 3}}{=} \frac{c}{d} \cdot \frac{a}{b}$$

17. Explain why each step is true.

Step 1 general formula for fraction multiplication

Step 2 multiplying whole numbers is commutative

Step 3 general formula for fraction multiplication

**Associative Property of Multiplication**

$\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}$

Look at the proof of the Associative Property below.

$$\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) \underset{\text{Step 1}}{=} \frac{a}{b} \cdot \frac{c \cdot e}{d \cdot f} \underset{\text{Step 2}}{=} \frac{a \cdot (c \cdot e)}{b \cdot (d \cdot f)} \underset{\text{Step 3}}{=} \frac{(a \cdot c) \cdot e}{(b \cdot d) \cdot f} \underset{\text{Step 4}}{=} \frac{a \cdot c}{b \cdot d} \cdot \frac{e}{f} \underset{\text{Step 5}}{=} \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}$$

18. Explain why each step is true.

Step 1 general formula for fraction multiplication

Step 2 general formula for fraction multiplication

Step 3 multiplying whole numbers is associative

Step 4 general formula for fraction multiplication

Step 5 general formula for fraction multiplication

## Multiplication as Scaling

Lesson **9**

**Predicting Products** At various times throughout the unit, students are asked to predict how a product will compare to one of the factors. This idea is the focus of Lesson 9. Students predict the size of a product relative to the size of one factor based on the size of the other factor.

**Predict whether the product will be greater than, less than, or equal to the second factor. Then compute the product.**

Predictions may vary.

1. $\frac{2}{5} \cdot \frac{3}{4} = x$

   Predict: $x < \frac{3}{4}$

   Compute: $x = \frac{3}{10}$

2. $\frac{6}{6} \cdot \frac{3}{4} = x$

   Predict: $x = \frac{3}{4}$

   Compute: $x = \frac{3}{4}$

3. $1\frac{3}{7} \cdot \frac{3}{4} = x$

   Predict: $x > \frac{3}{4}$

   Compute: $x = 1\frac{1}{14}$

Students make the following generalizations.

- Multiplying a number *n* by a factor less than 1 gives a product less than *n*.
- Multiplying a number *n* by a factor equal to 1 gives a product equal to *n*.
- Multiplying a number *n* by a factor greater than 1 gives a product greater than *n*.

The second point above indicates that we can change a fraction to an equivalent fraction by multiplying it by 1 in the form $\frac{n}{n}$. This is equivalent to multiplying both the numerator and denominator by *n*, the method students learned in Unit 1. For example:

$$\frac{4}{7} = \frac{4}{7} \cdot \frac{3}{3} = \frac{12}{21} \qquad \frac{4}{7} = \frac{4 \cdot 3}{7 \cdot 3} = \frac{12}{21}$$

**Predicting Solutions to Word Problems** Students then solve multiplication word problems, like the one below, where they must predict the size of the answer before computing.

16. A box of granola weighs 18 ounces. A box of corn flakes weighs $\frac{7}{9}$ as much as the granola.

    Do the corn flakes weigh more or less than 18 ounces?

    less

    How much do the corn flakes weigh?

    14 ounces

### *from* THE PROGRESSIONS FOR THE COMMON CORE STATE STANDARDS ON NUMBER AND OPERATIONS—FRACTIONS

**Multiplication as Scaling** Previous work with multiplication by whole numbers enables students to see multiplication by numbers bigger than 1 as producing a larger quantity, as when a recipe is doubled, for example. Grade 5 work with multiplying by unit fractions, and interpreting fractions in terms of division, enables students to see that multiplying a quantity by a number smaller than 1 produces a smaller quantity, as when the budget of a large state university is multiplied by $\frac{1}{2}$, for example.

## Dividing with Whole Numbers and Unit Fractions

Lessons **10** **11** **12** **13**

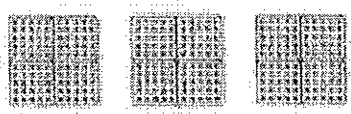
**Cases of Division** Students learn three cases of division with fractions in Grade 5.

- Dividing a whole number by a whole number in cases where the result is a fraction
- Dividing a whole number by a unit fraction
- Dividing a unit fraction by a whole number

**Divide a Whole Number by a Whole Number** A real world situation is presented to help students understand that for any two whole numbers $a$ and $b$ that are greater than or equal to 1, $a \div b = a \cdot \frac{1}{b} = \frac{a}{b}$.

There are 4 people in the Walton family, but there are only 3 waffles. How can the Waltons share the waffles equally?

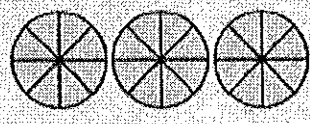
Divide each waffle into 4 pieces.



Each person's share of one waffle is $\frac{1}{4}$. Since there are 3 waffles, each person gets 3 of the $\frac{1}{4}$s, or $\frac{3}{4}$ of a whole waffle.

$3 \div 4 = 3 \cdot \frac{1}{4} = \frac{3}{4}$

**Divide a Whole Number by a Unit Fraction** Visual models are used to illustrate that for any whole number $w$ and any unit fraction $\frac{1}{d}$, $w \div \frac{1}{d} = w \cdot d$. For example, the diagram below is used to model $3 \div \frac{1}{8}$. Because there are 8 eighths in each 1 whole, there are $3 \cdot 8$ eighths in 3 wholes. That is $3 \div \frac{1}{8} = 3 \cdot 8 = 24$.



*from* THE PROGRESSIONS FOR THE COMMON CORE STATE STANDARDS ON NUMBER AND OPERATIONS—FRACTIONS

**Fractions and Division** In Grade 5, they connect fractions with division, understanding that $5 \div 3 = \frac{5}{3}$, or, more generally, $\frac{a}{b} = a \div b$, for whole numbers $a$ and $b$, with $b$ not equal to zero. They can explain this by working with their understanding of division as equal sharing.

*from* THE PROGRESSIONS FOR THE COMMON CORE STATE STANDARDS ON NUMBER AND OPERATIONS—FRACTIONS

**Divide a Whole Number by a Whole Number** Having seen that division of a whole number by a whole number, e.g. $5 \div 3$, is the same as multiplying the number by a unit fraction, $\frac{1}{3} \times 5$, they now extend the same reasoning to division of a unit fraction by a whole number, seeing for example that $\frac{1}{6} \div 3 = \frac{1}{6 \times 3} = \frac{1}{18}$.

Also, they reason that since there are 6 portions of $\frac{1}{6}$ in 1, there must be $3 \times 6$ in 3, and so $3 \div \frac{1}{6} = 3 \times 6 = 18$.

**Divide a Unit Fraction by a Whole Number** A real world situation and visual model are used to develop the idea of dividing a unit fraction by a whole number. Specifically, students consider the case of $\frac{1}{3} \div 5$. This can be interpreted as the question: If we divide $\frac{1}{3}$ into 5 parts, how big is each part? The diagram below shows $\frac{1}{3}$ divided into 5 parts:

$\frac{1}{3} \div 5$

To find the size of each part, we need to divide the other two thirds into five parts, which is like multiplying by $\frac{1}{5}$. Each part is $\frac{1}{15}$. So, $\frac{1}{3} \div 5 = \frac{1}{15}$.

$\frac{1}{3} \div 5 = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$

In general, for any unit fraction $\frac{1}{d}$ and any whole number $w$ that is greater than or equal to 1, $\frac{1}{d} \div w = \frac{1}{d} \cdot \frac{1}{w} = \frac{1}{d \cdot w}$

**Real World Problems** In Lesson 11, students describe real world situations and draw diagrams to represent division problems. They also solve division word problems by writing and solving equations.

In Lesson 12, students solve mixed multiplication and division word problems. For some problems, students must determine the operation and predict the size of the result before computing.

Lesson 13 reviews all four fraction operations.

## Focus on Mathematical Practices

Lesson 14

The Standards for Mathematical Practice are included in every lesson of this unit. However, there is an additional lesson that focuses on all eight Mathematical Practices. In this lesson, students explore a pie chart showing the instruments in a marching band. The chart shows what fraction of the band is made up of each type of instrument.

*from* **THE PROGRESSIONS FOR THE COMMON CORE STATE STANDARDS ON NUMBER AND OPERATIONS—FRACTIONS**

**Real World Problems** Students use story problems to make sense of division problems:

> How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?